

STABILITY OF MRI TURBULENT ACCRETION DISKS

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ABSTRACT

Based on the characteristics of the magnetorotational instability (MRI) and the MRI-driven turbulence, we construct a steady model for a geometrically thin disk using "non-standard" α -prescription. The efficiency of the angular momentum transport depends on the magnetic Prandtl number, $Pm = \nu/\eta$, where ν and η are the microscopic viscous and magnetic diffusivities. In our disk model, Shakura-Sunyaev's α -parameter has a power-law dependence on the magnetic Prandtl number, that is $\alpha \propto Pm^\delta$ where δ is the constant power-law index. Adopting Spitzer's microscopic diffusivities, the magnetic Prandtl number becomes a decreasing function of the disk radius when $\delta > 0$. The transport efficiency of the angular momentum and the viscous heating rate are thus smaller in the outer part of the disk, while these are impacted by the size of index δ . We find that the disk becomes more unstable to the gravitational instability for a larger value of index δ . The most remarkable feature of our disk model is that the thermal and secular instabilities can grow in its middle part even if the radiation pressure is negligibly small in the condition $\delta > 2/3$. In the realistic disk systems, it would be difficult to maintain the steady mass accretion state unless the Pm -dependence of MRI-driven turbulence is relatively weak.

Subject headings: accretion, accretion disks — instabilities — magnetic fields — MHD

1. INTRODUCTION

The physical mechanism to transport angular momentum in accretion disks is an important unsettled issue in astrophysics. It has been extensively studied for decades using theoretical and numerical procedures. The main purpose of these studies is to find the efficient transport process of the angular momentum to account for the powerful mass accretion deeply associated with the release of the gravitational energy observed in astrophysical disk systems (King et al. 2007).

Shakura & Sunyaev (1973) propose a pioneering disk model for describing the steady mass accretion, in which the efficiency of the angular momentum transport is parametrized and represented by a dimensionless viscosity parameter $\alpha = -t_{r\phi}/p$, where $t_{r\phi}$ is the $r\phi$ -component of the stress tensor, and p is the total pressure. The phenomenological parameter α introduced in the disk model should be determined experimentally. This disk model is widely accepted today as the standard, and referred to as "Standard Shakura-Sunyaev Model" (Kato et al. 1998, hereafter, we simply call it the ' α -model').

It is pointed out by King et al. (2007) from the observational point of view that the viscosity parameter α should be at the level $\alpha \simeq 0.1$ in order to explain the powerful outbursts from disk systems, such as dwarf novae, X-ray transients, and a fully ionized part of the disk in Active Galactic Nucleus (AGNs). Since the required level for the viscosity parameter α is much larger than that resulting from the microscopic molecular viscosity, the turbulent viscosity becomes the most promising candidate for governing the angular momentum transport in astrophysical disk systems (see also Hartmann et al. 1998; Smak 1999; Dubus et al. 2001; Starling et al. 2004; Lodato & Clarke 2004).

The Keplerian disks are known to be linearly stable to the shear instability (Rayleigh instability) which could power the

hydrodynamic turbulence. Even when the non-linear perturbations are imposed, the Keplerian disks can never reach the highly turbulent state in the Rayleigh stable condition (Balbus et al. 1996). Numerical studies in addition show that the other hydrodynamic instabilities, such as convective and Papaloizou-Pringle instabilities, do not play a central role in inducing the efficient angular momentum transport in the disk systems (Papaloizou & Pringle 1984; Blaes & Hawley 1988; Stone & Balbus 1996; Balbus & Hawley 1998; Johnson & Gammie 2006).

Balbus & Hawley (1991) focus, for the first time, on the magneto-rotational instability (MRI) which is originally found by Velikov (1959) and Chandrasekhar (1960) as a candidate for operating the magnetohydrodynamic (MHD) turbulence in the disk systems. It has essentially a local nature and destabilizes weakly magnetized differentially rotating systems with a negative shear rate $q \equiv d \ln \Omega / d \ln r$, where Ω is the angular velocity and r is the cylindrical radius. The non-linear MHD turbulence driven by the MRI has been broadly investigated as the leading mechanism for the turbulent angular momentum transport in disk systems using state-of-the-art numerical techniques and huge computer facilities for the MHD simulations (Hawley & Balbus 1992; Hawley 1995; Sano et al. 2004; Fromang et al. 2007). The physical properties of the MRI-driven turbulence, however, have not yet been confirmed completely even today.

A recent remarkable finding in the MRI studies is the dependence of the transport efficiency of the MRI-driven turbulence on the magnetic Prandtl number Pm (Lesur & Longaretti 2007; Fromang et al. 2007; Simon & Hawley 2009), where $Pm = \nu/\eta$, ν is the viscosity and η is the magnetic diffusivity. The local shearing box simulations explicitly taking into account the microscopic kinematic viscosity and resistivity suggest that the total turbulent stress (sum of Maxwell and Reynolds stresses) depends on the magnetic Prandtl number. Especially in the system with the non-zero net magnetic flux, the turbulent stress and the viscosity parameter α resulting from the MRI-driven turbulence would increase with the magnetic Prandtl

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number, that is $\alpha \propto Pm^\delta$, where δ is the power-law index with positive value (Fromang et al. 2007; Simon & Hawley 2009).

It should be stressed that the viscous effect on the non-linear MRI-turbulence is seemingly contradictory to that on the linear growth of the MRI (Masada & Sano 2008; Pessah & Chan 2008). Surprisingly, the viscosity can boost the transport efficiency of the MRI-driven turbulence although it suppresses the growth of the MRI in its linear evolutionary regime. The physical mechanism for the viscous boosting of the MRI-driven turbulence is a central issue in the current MRI studies and remains to be solved to draw a complete physical picture of the angular momentum transport in the accretion disks (Longaretti & Lesur 2010).

The impact of the microscopic diffusivities on the MRI-driven turbulence should be essential not only in understanding the nature of the MRI itself, but also in applying it to the realistic astrophysical disk systems. Assuming the standard α -disk, the size of the magnetic Prandtl number drastically changes depending on the disk radius. Balbus & Henri (2008) suggest that the magnetic Prandtl number Pm is typically larger in the inner disk when we adopt Spitzer's values for evaluating the microscopic diffusivities (Spitzer 1962). Hence, the transport efficiency of the angular momentum should be affected by the magnetic Prandtl number and become a function of the disk radius when the viscosity parameter follows the relation $\alpha \propto Pm^\delta$ which is implied from the recent local shearing box simulations (Lesur & Longaretti 2007; Simon & Hawley 2009).

In this paper, we construct a steady model for geometrically thin disk taking account of the "non-standard" description of the viscosity parameter which depends on the magnetic Prandtl number in the form $\alpha \propto Pm^\delta$. This should be consistent treatment of the turbulent viscosity with the results from recent local shearing box simulations (Lesur & Longaretti 2007; Simon & Hawley 2009). A central difference between the work done by Balbus & Henri (2008) and ours is to include the structural change of the disk depending on the size of the magnetic Prandtl number Pm in our model consistently.

This paper is organized as follows: The steady disk model with the non-standard description of the viscosity parameter $\alpha \propto Pm^\delta$ is constructed in § 2. We then describe the basic properties of our disk model. In § 3, we investigate the stability of our steady disk model to the gravitational, thermal, and secular instabilities. The dependence of the stability criteria on the magnetic Prandtl number (or index δ) is our interest in this section. Finally, we discuss the application of our model to the realistic astrophysical disk system and summarize our findings in § 4.

2. GEOMETRICALLY THIN DISK WITH NON-STANDARD α -PRESCRIPTION

We construct a steady disk model consistently combining the properties of the MRI-driven turbulence recently found by numerical studies (Lesur & Longaretti 2007; Simon & Hawley 2009). The fundamental assumption adopted in this work is that the viscosity parameter α depends on the disk radius and reflects the transport properties of the MRI-driven turbulence. Also we assumed that the disk heating is local in which the variation of the thermal energy responds to the turbulent energy dissipation instantaneously (Balbus & Papaloizou 1999). This assumption is consistent with the recent numerical work of the MRI-driven turbulence (Sano & Inutsuka 2001; Simon et al. 2009; Guan et al. 2009).

According to the recent numerical studies, the efficiency of the angular momentum transport sustained by the MRI-driven turbulence is regulated by the magnetic Prandtl number which characterizes the system. Using the viscosity parameter α , the transport efficiency of the MRI-driven turbulence can be represented by, as is described in § 1,

$$\alpha = \alpha_0 Pm^\delta, \quad (1)$$

(Lesur & Longaretti 2007; Simon & Hawley 2009), where α_0 is the normalization parameter which controls the size of α to be smaller than unity. Note that this relation reduces to the classical standard α -model by taking $\delta = 0$.

For providing the magnetic Prandtl number, we adopt the Spitzers' values for the microscopic kinematic viscosity ν and the electron resistivity η by assuming the fully ionized gas. This assumption is reasonable for the disk of the X-ray binaries and for the inner part of the disk in AGNs. Then ν and η are expressed as

$$\nu = 1.6 \times 10^{-15} \rho^{-1} T^{\frac{5}{2}} (\ln \Lambda_{HH})^{-1}, \quad (2)$$

and

$$\eta = 5.55 \times 10^{11} T^{-\frac{3}{2}} \ln \Lambda_{eH}, \quad (3)$$

where ρ is the disk mass density, and T is the disk temperature (Spitzer 1962). The Coulomb logarithms for proton-proton and electron-proton scatterings are represented above by $\ln \Lambda_{HH}$ and $\ln \Lambda_{eH}$, respectively. The magnetic Prandtl number is then described as

$$Pm = \left(\frac{T}{4.2 \times 10^4 \text{ K}} \right)^4 \left(\frac{10^{14} \text{ cm}^{-3}}{nl} \right), \quad (4)$$

where n is the number density and $l = \ln \Lambda_{eH} \ln \Lambda_{HH}$ is the product of the Coulomb logarithms and fixed to be $l = 40$ in the following (see Balbus & Henri 2008).

The governing equations are the conventional used in the classical standard α -model:

- Mass conservation equation

$$\dot{M} = -2\pi r v_r \Sigma. \quad (5)$$

- Force balancing equation (Keplerian rotation)

$$\Omega = \Omega_K \equiv \sqrt{\frac{GM}{r^3}}. \quad (6)$$

- Angular momentum conservation

$$\nu_t \Sigma = \frac{\dot{M}}{3\pi} \left(1 - \sqrt{\frac{R_{in}}{r}} \right) \equiv \frac{\dot{M}}{3\pi} f. \quad (7)$$

- Energy balancing equation

$$\frac{9}{4} \nu_t \Sigma \Omega^2 = \frac{32\sigma T^4}{3\tau}. \quad (8)$$

- Hydrostatic balance in the vertical direction

$$H = \frac{c_s}{\Omega}. \quad (9)$$

where Ω_K is the Keplerian angular velocity. The physical parameters M , \dot{M} , v_r , c_s , ν_t , H , R_{in} and τ are the mass of the central objects, the mass accretion rate, the radial velocity, the sound speed, the turbulent viscosity, the vertical scale height of the disk, the inner radius of the disk, and the optical depth

in the vertical direction. Here Σ is the surface density defined by $\Sigma = 2\rho H$. The physical constants G , c , and σ are the gravitational constant, the speed of light, and the Stefan-Boltzmann constant respectively.

In this paper, we naively assume that the viscosity parameter α is related to the turbulent viscosity by a relation,

$$\alpha p = \frac{3}{2} \rho \nu_t \Omega, \quad (10)$$

(but, see Sano et al. 2004; Pessah et al. 2007, and § 4 in this paper). Note that there is a clear distinction between the turbulent viscosity ν_t and the kinematic viscosity ν . In our model, the turbulent viscosity ν_t is determined by the MRI-driven turbulence which controls the accretion dynamics. The transport efficiency due to the MRI-driven turbulence is regulated via the relation (1) by the microscopic kinematic viscosity ν . The kinematic viscosity affects the global nature indirectly by regulating the MRI-driven turbulence.

The opacity of the disk κ , which relates to the optical depth τ as $\kappa = 2\tau/\Sigma$, is given by

$$\begin{aligned} \kappa &= \kappa_{\text{es}} + \kappa_{\text{ff}} \\ &= \kappa_{\text{es}} + \kappa_0 \rho T^{-7/2}. \end{aligned} \quad (11)$$

where $\kappa_{\text{es}} \simeq 0.4 \text{ cm}^2 \text{ g}^{-1}$ is the opacity due to the electron scattering and κ_{ff} is that contributed from the free-free absorption with $\kappa_0 \simeq 6.4 \times 10^{22}$ in *cgs* unit.

For the closure of the system, we need the equation of state for the composite gas of the photon and baryon,

$$p = \frac{\rho}{\mu m_p} k_B T + \frac{4\sigma T^4}{3c}. \quad (12)$$

where m_p is the proton mass, k_B is the Boltzmann constant and μ is the mean molecular weight providing $\mu = 0.5$ throughout this paper.

Solving coupled equations (1), (4)-(12) with the zero stress boundary condition at $r = R_{\text{in}}$ (see, equation 7, or, Novikov & Thorne 1973), we can obtain the local disk structures for three characteristic regimes. In the inner part of the disk, the radiation pressure dominates the gas pressure. In contrast, the gas pressure becomes dominant in the middle and outer regions of the disk in our model. The difference between the middle and outer regions is the source responsible for the opacity. The electron scattering plays a crucial role in determining the opacity in the middle region. The outer structure of the disk depends mainly on the opacity contributed by the free-free absorption (see, also standard α -model in Shakura & Sunyaev 1973).

The steady disk structure obtained here is then characterized by four dimensionless parameters, $x = r/r_s$, $m = M/M_\odot$, $\dot{m} = \dot{M}c^2/L_E$ and $\delta = \log(\alpha/\alpha_0)/\log Pm$, where r_s is the Schwarzschild radius, M_\odot is the solar mass, and L_E is the Eddington luminosity, respectively. The innermost radius R_{in} is assumed as $R_{\text{in}} = 3r_s$ in the following.

Figures 1 and 2 show the radial profiles of the magnetic Prandtl number (left top), α (right top), the radial velocity in unit of c (left middle), the scale height (right middle), the surface density (left bottom), and the disk temperature (right bottom) for the models $m = 10$ (Fig. 1: the model X-ray binaries) and $m = 10^8$ (Fig. 2: the model AGNs), respectively (c.f., Balbus & Henri 2008). The mass accretion rate is fixed as the Eddington one, that is $\dot{m} = 1$ in both figures. The normalization parameter α_0 is an arbitrary constant and is determined for the viscosity parameter α being unity at its max-

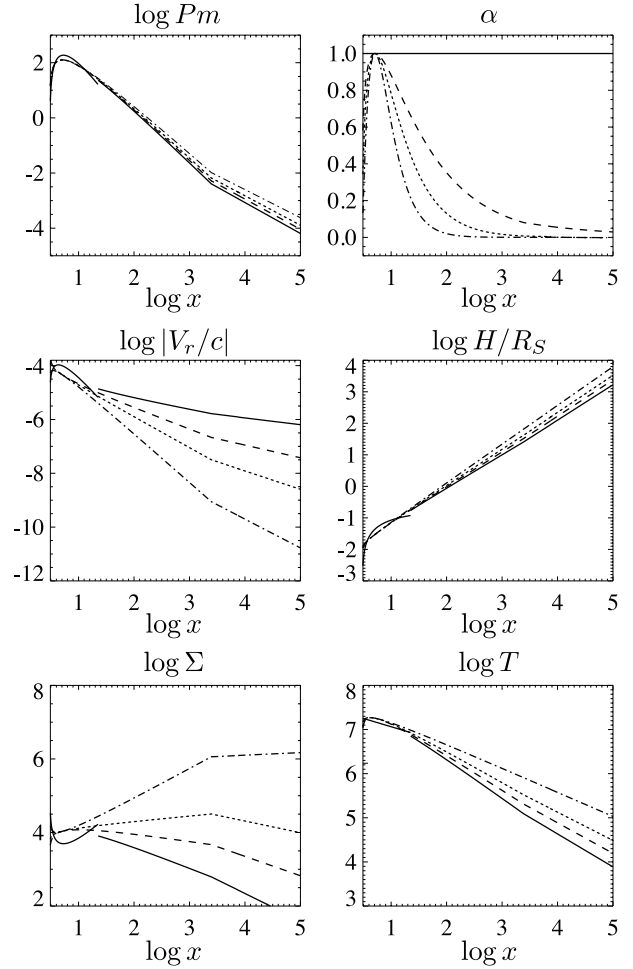


FIG. 1.— Radial profiles of the magnetic Prandtl number (left top), α (right top), the radial velocity (left middle), the scale height (right middle), the surface density (left bottom), and the disk temperature (right bottom). α_0 is determined so that the maximum of α is unity. Thick solid curves denote the solutions for the conventional standard disk model, while dashed, dotted, and dash-dotted ones do those for $\delta = 0.25, 0.5, 1$, respectively. The other parameters are $m = 10$, $\dot{m} = 1$.

imum. Solid curves denote the solutions for $\delta = 0$ (conventional standard α -model), while the dashed, dotted, and dash-dotted curves are for the cases $\delta = 0.25, 0.5, 1.0$ respectively.

We find that the magnetic Prandtl number is smaller than unity in the outer part of the disk, while it increases with decreasing the radius. Around the middle part of the disk, there is a critical radius r_c where the magnetic Prandtl number exceeds unity. It exists around $r_c \simeq 100r_s$ for both models. This is qualitatively consistent with the result obtained in Balbus & Henri (2008). The kinematic viscosity increases with the disk temperature and the magnetic diffusivity decreases with it. Since the disk temperature is a decreasing function of the disk radius, the magnetic Prandtl number varies sharply with the disk temperature and becomes lower and lower with the increase in the disk radius [see equation (3)] except the inner disk. It would be important to note that the magnetic Prandtl number is insensitive to the index δ although the disk temperature largely changes depending on it in the outer part of the disk. This is because the surface density varies more rapidly with the disk radius compared to the temperature not for violating the energy conservation law [see equation (8)].

The efficiency of the angular momentum transport and the viscous heating, which can be represented by the viscosity pa-

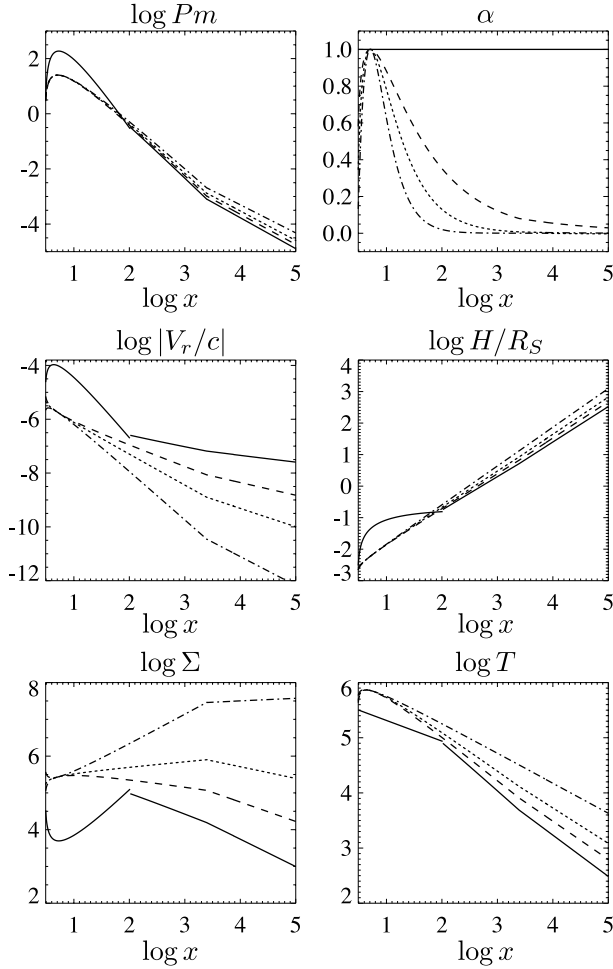


FIG. 2.— Radial profiles of the magnetic Prandtl number (left top), α (right top), the radial velocity (left middle), the scale height (right middle), the surface density (left bottom), and the temperature (right bottom). α_0 is determined so that the maximum of α is unity. Thick solid curves denote the solutions for the conventional standard disk model ($\delta = 0$), while dashed, dotted, and dash-dotted ones do those for $\delta = 0.25, 0.5, 1$, respectively. The other parameters are $m = 10^8$, $\dot{m} = 1$.

parameter α , must be lower in the outer part of the disk in our model. In addition, the parameter α strongly depends on the index δ . We stress here that, for the model with the larger index δ , the viscosity parameter α changes more steeply with radius. This tendency can be understood from equation (1) by considering that magnetic Prandtl number is insensitive to the index δ . The lower transport efficiency of the angular momentum results in the higher density and the lower accretion velocity in the outer part of the disk to ensure the constant mass accretion rate [see equation (5)]. The drastic structural change in the middle and outer regions of the disk is a natural consequence of the Pm -dependence of the viscosity parameter α .

We mention the physical behaviors of the inner disk. In our model, there exists a maximum of the parameter α at around the innermost radius R_{in} . The disk structure is drastically changed across this point. Since we adopted the zero stress boundary condition at the inner radius (Novikov & Thorne 1973), the gaseous material cannot rotate with the Keplerian velocity inside R_{in} . The gas would thus accrete to the central object while being less affected by the turbulent viscous heating in such a region. This results in the low disk temperature and then the low transport efficiency of the angular momen-

tum (plunging region). This should be the reason why there is an inflection point in the disk structure in our model.

3. STABILITY OF DISK MODEL

The inclusion of the Pm -dependence in the viscosity parameter α makes drastic changes in the disk structure. We investigate the linear stability of the disk model with "non-standard" α -prescription to the gravitational, thermal, and secular instabilities.

3.1. Gravitational Instability

As is shown in § 2, the surface density in our disk model is larger for a larger δ , especially in the outer region, than that in the standard α -disk model. This is because the transport efficiency of the angular momentum becomes lower in the outer disk when we consider the Pm -dependence of the viscosity parameter. The larger amount of the gaseous material must be distributed there for the larger value of the index δ . It would be thus affected more by the self-gravity in our disk model than the standard α -model.

We restrict our attention to the disk whose potential is dominated by the central object and whose rotation curve is therefore Keplerian. The gravitational instability to axisymmetric perturbations then sets in when the sound speed c_s , the rotation frequency Ω , and the surface density Σ satisfy

$$Q = \frac{\Omega c_s}{\pi G \Sigma} \lesssim 1, \quad (13)$$

(Toomre 1964; Goldreich & Lynden-Bell 1965). The instability condition (13) can be rewritten, for a disk with scale height $H \simeq c_s/\Omega$ around a central object of mass M ,

$$M_{\text{disk}} \gtrsim \frac{H}{r} M, \quad (14)$$

where $M_{\text{disk}} = \pi r^2 \Sigma$ (see Johnson & Gammie 2003, for details). The disk becomes unstable to the gravitational instability when the self-gravity of the gaseous material overcomes the gravitational force provided by the central object acting upon it.

Figures 3 and 4 show the radial profiles of Toomre's Q -value for the models $m = 10$ and $m = 10^8$. Solid, dashed, dotted, and dash-dotted curves denote the cases $\delta = 0, 0.25, 0.5, 1.0$, respectively. The mass accretion rate is fixed as $\dot{m} = 1$ in both figures. It is found from these figures that Toomre's Q -value is smaller for the model with the larger index δ . This is because the larger amount of gaseous material with the slower accretion velocity should be in the outer regions for the larger index δ . Hence, the disk becomes more unstable to the gravitational instability for the models with the larger index δ .

When $m = 10$, Toomre's Q -value becomes smaller than unity in the outer part where $r \gtrsim 10^8 r_s$. The X-ray binaries, which are the binary system of the stellar mass black hole and the companion stellar object, would be gravitationally stable because the typical size of these systems would be much smaller than $10^8 r_s$. In contrast, for the model $m = 10^8$ (AGNs), Toomre's Q -value is smaller than unity in the region $r \gtrsim 10^2 - 10^3 r_s$. This indicates that the geometrically thin disk characterizing AGN systems is gravitationally unstable, not only in our model, but also in the conventional α model. The Pm -dependence of the viscosity parameter, which is the assumption based on the recent MRI studies, enhances the self-gravitational instability of the accretion disk because of the

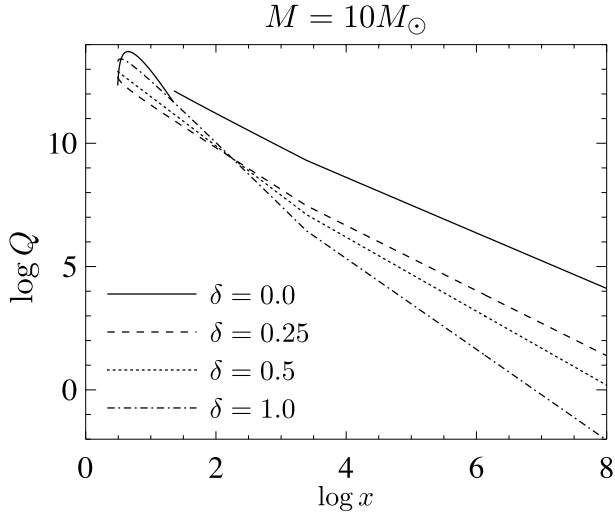


FIG. 3.— Radial profile of Toomre's Q -value for $m = 10$. Solid, dashed, dotted, and dash-dotted curves represent for $\delta = 0, 0.25, 0.5, 1.0$, respectively. lower transport efficiency of the angular momentum in the region characterized by the lower magnetic Prandtl number.

The self-gravity should play a significant role in enhancing the turbulent viscosity, and triggering the disk fragmentation (Bodenheimer et al. 1980; Shore & White 1982; Lin & Pringle 1987; Shlosman & Begelman 1987). Although we must take account of these self-gravity effects consistently in our governing equations to construct the disk model correctly capturing the AGNs disk system (Kozłowski et al. 1979; Mineshige & Umemura 1996), it is beyond the scope of this paper. The main purpose of this paper is to extend the standard α -disk to the model with Pm -dependence of the viscosity parameter as the first step.

3.2. Thermal and Secular Instabilities

Equations (1) and (4) indicate that the lower disk temperature provides the lower transport efficiency of the angular momentum, and therefore the lower viscous heating rate. The radial variations of the viscous parameter associated with radially declining viscous heating would impact on the responses of the disk model to the thermal and secular instabilities (Balbus & Henri 2008).

Since the thermal timescale is much longer than the dynamical one, we can assume that the dynamical equilibrium is retained when we consider the thermal instability. In addition, we can assume that the surface density of the disk does not change in the thermal evolution time because the centrifugal force balances with gravitational force in the radial direction. Thus the gaseous motion caused by the temperature perturbation is restricted almost in the vertical direction. These assumptions can simplify the treatment of the thermal instability.

We consider the thermal equilibrium as the unperturbed state in the condition with the fixed surface density. When the disk temperature is slightly perturbed over that of the equilibrium state, the criterion for the thermal instability is given

$$\left. \frac{\partial(Q_{\text{vis}}^+ - Q_{\text{rad}}^-)}{\partial T} \right|_{\Sigma=\text{const}} > 0, \quad (15)$$

where

$$Q_{\text{vis}}^+ = -\frac{3}{2} \Omega T_{r\phi}, \quad (16)$$

$$Q_{\text{rad}}^- = \frac{32\sigma T^4}{3\tau}, \quad (17)$$

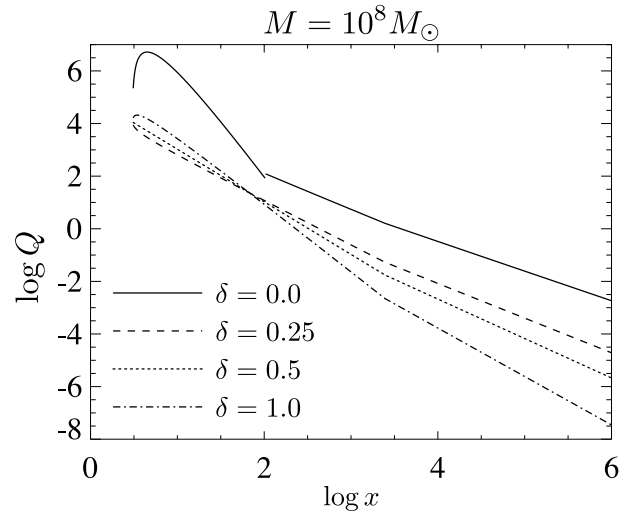


FIG. 4.— Radial profile of Toomre's Q -value for $m = 10^8$. Solid, dashed, dotted, and dash-dotted curves represent for $\delta = 0, 0.25, 0.5, 1.0$, respectively. (Shakura & Sunyaev 1973; Pringle 1976; Piran 1978). Here Q_{vis}^+ is the viscous heating rate (Pringle 1976), and Q_{rad}^- is the radiative cooling rate. The thermal equilibrium satisfies the condition $Q_{\text{vis}}^+ = Q_{\text{rad}}^-$. Note that the viscous heating and radiative cooling rates are both the increasing function of the disk temperature. The disk becomes thermally unstable when the increasing rate of the viscous heating overcomes that of the radiative cooling. Once the disk temperature is slightly increased from the equilibrium state, it is further enhanced by the positive feedback from the viscous heating which is more effective than the radiative cooling (Kato et al. 1998).

The secular instability is a phenomenon resulting from the spatial modulation of the accretion rate. The typical timescale in which the instability grows is thus the viscous timescale, which is much longer than the thermal and dynamical ones in the geometrically thin disk. Thus we can assume that the state is in thermal and dynamical equilibria, so that the thermal and oscillatory modes can be filtered out. Then the criterion for the secular instability is

$$-\left. \frac{\partial T_{r\phi}}{\partial \Sigma} \right|_{Q_{\text{vis}}^+ = Q_{\text{rad}}^-} < 0, \quad (18)$$

(Lightman & Eardley 1974; Kato et al. 1998) where $T_{r\phi} = -2\alpha p H$ is the $r\phi$ -component of the turbulent stress tensor. This means that the instability sets in when the turbulent stress decreases with increasing the surface density. Once the surface density is perturbed positively from the dynamical equilibrium state, the turbulent stress and the efficiency of the angular momentum transport decreases. The perturbation of the surface density is thus further enhanced by the positive feedback (\sim negative diffusion).

For the simple treatment of two types of instabilities above, we introduce dimensionless parameters β and γ . The ratio of the gas pressure p_{gas} to the total pressure p , which is the sum of the gas pressure and the radiation pressure p_{rad} , can provide the parameter β (Piran 1978)

$$\beta \equiv \frac{p_{\text{gas}}}{p}. \quad (19)$$

The $r\phi$ component of the stress tensor $T_{r\phi}$ has a general form

$$T_{r\phi} = -2\alpha p_{\text{gas}}^\gamma p^{1-\gamma} H. \quad (20)$$

From these definition, the parameters β and γ range $0 \leq \beta, \gamma \leq 1$. The solution of $\gamma = 0$ corresponds to that of the

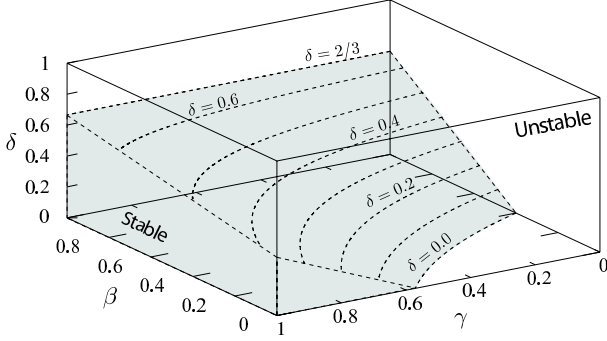


FIG. 5.— Thermal and secular unstable domain on the parameter space of β – γ – δ . We assume electron-electron scattering for the main opacity. The shaded domain shows the stable region. Shakura-Sunyaev disk, and $\gamma = 1$ to that where $T_{r\phi}$ is proportional to the gas pressure instead of the total pressure.

By adding the small-amplitude perturbations to the systems described by set of equations (4)–(12) including the Pm -dependence of the viscosity parameter α with the assumptions discussed above, we obtain the criteria for the thermal and secular instabilities with the parameters δ , β , and γ , (see, appendix for derivation).

When we assume the electron scattering mainly contributes to the opacity, we can combine the criteria for the thermal and secular instabilities to a single condition (Kato et al. 1998) as

$$4 - 10\beta - 7\gamma(1 - \beta) + \delta(8 + \beta) > 0. \quad (21)$$

Note that the α -prescription depending on Pm is adopted here instead of the conventional approach. Neglecting the last term resulting from the Pm -dependence of the viscous parameter, equation (21) reduces to the criterion of thermal and secular instabilities for the standard α -model. Since the last term is always positive for the case $\delta \geq 0$, our disk model with 'non-standard' α -parameter should be more destabilized when the transport efficiency of the angular momentum by the MRI-driven turbulence depends more strongly on the magnetic Prandtl number.

Figure 5 shows the stability criterion for the thermal and secular instabilities in the parameter space of β – γ – δ . The shaded domain indicates the stable region. The disk is unstable for the model with the smaller β and γ . This indicates that the instabilities are generally facilitated when the contribution of the radiation pressure to the total pressure becomes larger. It is characteristic of our disk model that the unstable domain extends for a larger δ , indicating that the disk is more unstable when the magnetic Prandtl number strongly affects the angular momentum transport in MRI-driven turbulence.

Considering the Shakura-Sunyaev type stress tensor ($\gamma = 0$), the condition for the instability is rewritten by

$$\beta < \frac{4 + 8\delta}{10 - \delta}. \quad (22)$$

The disk becomes unstable when $\beta < 0.4$ in the condition $\delta = 0$ (standard α -model). The unstable parameter space extends when $\delta > 0$. This can be understood intuitively from equations (1) and (4). Our model gives the relation $\alpha \propto \rho^{-\delta} T^{4\delta}$ and thus the viscous heating rate increases with the disk temperature when $\delta > 0$. Since the larger index δ provides more efficient viscous heating, it should play a significant role in enhancing the thermal instability. On the other hand, the transport efficiency of the angular momentum decreases with the density when $\delta > 0$. The positive feedback effect becomes larger for the model with the larger δ . Hence the increase of the index δ also enhances the secular instability.

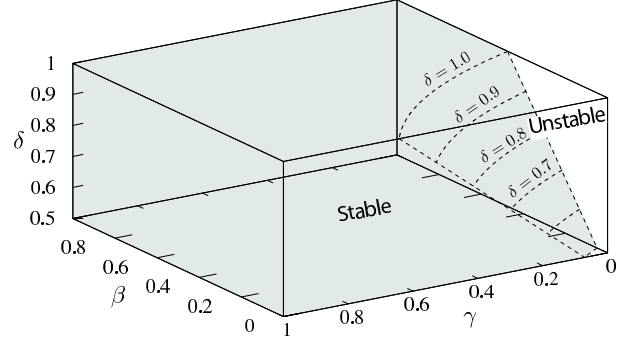


FIG. 6.— Thermal and secular unstable domain on the parameter space of β – γ – δ . We assume free-free opacity. The shaded domain shows the stable region.

We would like to stress here that the disk becomes unstable even if the radiation pressure is negligibly small ($\beta = 1$) when the condition $\delta > 2/3$ is satisfied. Recent local shearing box simulations for the MRI imply that the index δ ranges from 0.25 to unity (Lesur & Longaretti 2007; Simon & Hawley 2009). This suggests that the accretion disk sustained by the MRI-driven turbulence can become thermally and secularly unstable even when the gas pressure dominates the radiation pressure ($p_{\text{gas}} \gg p_{\text{rad}}$). This is an important property that does not appear in the standard α -model.

When the free-free emission is the dominant source of the opacity, the criteria for the thermal and secular instabilities are combined again to a single condition (see appendix for derivation)

$$7 + 21\beta + 14\gamma(1 - \beta) - 2\delta(8 + \beta) < 0. \quad (23)$$

Note that all terms except the last one take always positive. This suggests that the disk is always stable to the thermal and secular instabilities in the condition $\delta = 0$ when the free-free opacity becomes dominant. In contrast, the last term is always negative when $\delta > 0$. The outer region of our model then tends to be unstable to these instabilities even when the opacity of the disk is mainly contributed from the free-free absorption.

Figure 6 demonstrates the stability criterion for the thermal and secular instabilities in the parameter space of β – γ – δ . The shaded region indicates the stable domain. It is found that the parameter space for the disk being unstable to these instabilities extends with increasing the index δ . It is important that the disk becomes unstable to these instabilities in the case $\delta > 0$ although it is thermally and secularly stable in the whole parameter domain when $\delta = 0$.

The Shakura-Sunyaev type stress tensor ($\gamma = 0$) reduces the condition (23) to

$$\beta < -\frac{7 - 16\delta}{21 - 2\delta}. \quad (24)$$

The conventional α -disk ($\delta = 0$) is always stable because β should be positive. In the case $\delta = 0$, the radiative cooling rate Q_{rad}^- more steeply responds to the disk temperature than the viscous heating rate Q_{vis}^+ . The thermal instability is thus suppressed by the negative feedback resulting from the strong radiative cooling. When the viscosity parameter α depends on the magnetic Prandtl number, the viscous heating rate also becomes the function of the disk radius. Then the positive feedback effect is amplified and the disk becomes unstable because the viscous heating rate increases with the disk temperature more steeply than the radiative cooling rate. Since the denominator of equation (24) has a positive value according to the recent local shearing box simulations, the condition

(24) reduces to $\beta < 16\delta - 7$. Then the disk can be thermally and secularly unstable in the case $\delta > 7/16$ when the radiation pressure dominates the gas pressure ($\beta = 0$) with Shakura-Sunyaev type stress tensor ($\gamma = 0$).

When the gas pressure dominates the radiation pressure ($\beta = 1$), the criterion for the thermal and secular instabilities is given as $\delta > 14/9$ from equation (23). Thus the outer part of the accretion disks can be unstable to these instabilities when the efficiency of the angular momentum transport and the viscous heating rate are strongly controlled by the magnetic Prandtl number.

4. SUMMARY AND DISCUSSION

We construct a disk model with the non-standard α -prescription depending on the magnetic Prandtl number $\alpha \propto Pm^\delta$ according to the recent numerical studies for the MRI. The magnetic Prandtl number Pm evaluated from the Spitzer's value is a decreasing function with disk radius and it becomes smaller than unity for the region $r \gtrsim 100r_s$ (Balbus & Henri 2008).

As a result of Pm -dependence of the viscosity parameter α , the transport efficiency of the angular momentum becomes lower in the outer part of the disk in the model with larger value of the index δ . Since the accretion velocity decreases with δ due to the inefficient angular momentum transport, there should be a large amount of gaseous materials in the outer part of the disk. The self-gravity enhanced as a result of the Pm -dependence of α -parameter would make the disk more unstable to the self-gravitational instability especially for the AGN disks in the region $r \gtrsim 10^2 - 10^3 r_s$ (see Paczynski 1978; Kozłowski et al. 1979; Fukue & Sakamoto 1992; Mineshige & Umemura 1996, for the case with standard α -prescription).

Since our model gives the relation $\alpha \propto \rho^{-\delta} T^{4\delta}$, the viscous heating rate increases with the disk temperature when $\delta > 0$. This suggests that the positive temperature perturbation enhances the viscous heating rate and thus the positive feedback to the thermal instability. On the other hand, the transport efficiency of the angular momentum decreases with increasing the gas density when $\delta > 0$. This works in the same way as the negative diffusion and also enhances the positive feedback to the secular instability.

It is the most remarkable feature of our disk model that the thermal and secular instabilities can grow in its middle part even if the radiation pressure is negligibly small in the condition $\delta > 2/3$. Also the outer part of the disk can be unstable to thermal and secular instabilities when $\delta > 7/16$ (radiation pressure dominant with Shakura-Sunyaev type stress tensor, $\gamma = 0$) or $\delta > 14/9$ (gas pressure dominant). If the MRI plays an important role in operating the MHD-turbulence and the viscosity parameter α depends on the magnetic Prandtl number, the steady mass accretion would not be maintained due to the growth of these instabilities in the geometrically thin disk. The rapid transition from the steady accretion phase to the non-steady state should be considered to enable us to understand more deeply the origin of the light curve observed from the realistic astrophysical disk system.

In our model, we assumed that the turbulent stress tensor is just proportional to the pressure according to the standard α -disk model (see, equation 20). On the other hand, recent local shearing box simulations for the MRI imply that the pressure dependence of the turbulent viscosity might be weak compared to that assumed in the standard α -disk model, that is $T_{r\phi} \propto p^\epsilon$ where the index ϵ takes a value in the range

$1/6 \lesssim \epsilon \lesssim 1/4$ (Sano et al. 2004; Pessah et al. 2007).

When the pressure dependence of the turbulent viscosity is weaker than that assumed in the standard α -disk, the viscous heating rate is expected to become insensitive to the temperature perturbation. The disk thus becomes more stable to the thermal instability. Additionally, the secular instability should also be suppressed because the turbulent stress would then respond weakly to the density perturbation.

Now let us derive the stability condition of disks when the turbulent stress varies weakly depending on the pressure as is implied from the recent numerical studies. Using the prescription $T_{r\phi} = -2\alpha p_{\text{gas}}^\gamma p^\epsilon H$ [note that this equation reduces equation (20) when $\epsilon = 1 - \gamma$] and assuming the electron scattering as the dominant opacity source, the instability conditions are given by (A7) for the thermal instability, and (A12) for the secular instability (see Appendix for the details of derivation). Conditions (A7) and (A12) can commonly yield, by taking $\beta = 1$ (gas pressure dominant),

$$\delta > \frac{7 - \gamma - \epsilon}{9}, \quad (25)$$

Here we assume $\delta \geq 0$. This equation indicates that the weak dependence of the viscosity parameter on the pressure (i.e., $|\gamma|, |\epsilon| \ll 1$) has a stabilization effect on the thermal and secular instabilities.

For the model in which the free-free absorption is the main opacity source, the stability conditions for the thermal and secular instabilities can be obtained from (A9) and (A13), respectively. By taking $\beta = 1$, these conditions are combined to a single condition

$$\delta > \frac{15 - \gamma - \epsilon}{9}. \quad (26)$$

Here we assume $\delta \geq 0$. This also shows that the instabilities can be suppressed when the pressure dependence of the viscosity parameter is weaker than that assumed in the standard disk models.

It should be important to stress that even when the turbulent stress does not directly depend on the pressure (i.e., $\gamma = \epsilon = 0$), the disk can be unstable since the efficiency of the angular momentum transport α depends on the temperature and the density. In such case, the instabilities grow when $\delta > 7/9$ for the middle part of the disks and when $\delta > 5/3$ for the outer part of the disks. Since δ is predicted to be less than unity from the recent local simulations, the middle part of the disks can be unstable to these instabilities when the saturation levels of MRI-driven turbulence is strongly controlled by the magnetic Prandtl number.

Finally we discuss the validity of the assumption adopted in our disk model. Numerical studies for the MRI imply that transport properties of the MRI-driven turbulence depend on the magnetic Prandtl number as is given in equation (1).

This relation is verified only in the narrow parameter range $10^{-2} \lesssim Pm \lesssim 10$ and $10 \lesssim R_M \lesssim 10^4$ (Lesur & Longaretti 2007; Longaretti & Lesur 2010). Here $R_M = Hc_s/\eta$ is the magnetic Reynolds number. The range of Pm examined in numerical studies corresponds to the region $10r_s \lesssim r \lesssim 10^3 r_s$ in our model. The Pm -dependence of the viscosity parameter α is not well studied due to the computational issue in the parameter range characterizing the outer part of the disk model. In addition, Reynolds and magnetic Reynolds numbers are taken to be involuntarily larger values than the realistic values for the astrophysical ionized gas due to the limitation of the computational resources. If we suppose the realistic values of viscosity and resistivity, their related dissipation scales

become much smaller than those resolved in existing state-of-the-art numerical simulations. In order to facilitate the understanding of astrophysical accretion processes, we need to examine whether such small scale dissipation processes actually affect the dynamics of the accretion disks via the angular momentum transport and then the instabilities of the disk.

Additionally, it has not been confirmed whether the viscosity parameter is impacted by the magnetic Prandtl number when the microscopic diffusivities are the functions of the physical parameters such as the density and the temperature because the uniform diffusivities are assumed in the existing local shearing box simulations (Lesur & Longaretti 2007; Fromang et al. 2007; Simon & Hawley 2009). We need further systematic studies on the non-linear properties of the MRI in the vertically stratified disks with variable diffusive parameters and the realistic radiative cooling in order to verify the Pm -dependence of the turbulent viscosity parameter α .

Our model does not include just for simplicity the effect of magnetic fields explicitly (but takes account of the effect of the MHD turbulence through the non-standard α -prescription). The impacts of the magnetic field can modify the disk structure. Since the magnetic pressure contributes to the total pressure appeared in equation (20) and weakens the temperature dependence of the viscous heating, it might suppress the growth of gravitational, thermal and secular instabilities when the magnetic pressure dominates the gas and radiation pressures. This is beyond the scope of this paper but would be studied as our future work using numerical procedures.

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APPENDIX

CRITERIA FOR THE INSTABILITIES

In this appendix, we briefly summarize the derivation of the stability conditions shown by equations (21), (23), (25), and (26). Here we use a following general prescription for the stress tensor, instead of the form introduced by equation (20),

$$T_{r\phi} = -2\alpha p_{\text{gas}}^{\gamma} p^{\epsilon} H, \quad (\text{A1})$$

where ϵ is a constant. When $\epsilon = 1 - \gamma$, this equation reduces to equation (20). We hereafter describe the unperturbed and perturbed quantities by subscripts '0' and '1', respectively. Note that all the physical unperturbed quantities are assumed to be much larger than their associated perturbed quantities in the following.

We consider the stability criterion of the system to the thermal instability. As is discussed in § 3.2, we can assume that the dynamical equilibrium is maintained during the growth of the thermal instability since the dynamical time scale is much shorter than the thermal one. Also we can assume that the material motion resulting from a temperature variation occurs only in vertical direction since the force balances with the gravitational force in the radial direction. Then the surface density Σ is approximately unchanged during the growth of the thermal instability;

$$\Sigma = \text{const}, \quad (\text{A2})$$

while the hydrostatic balance is maintained (Kato et al. 1998).

The linearized form of equation (9) and the total pressure $p = p_{\text{gas}} + p_{\text{rad}}$, where p_{rad} is the radiation pressure, are

$$\frac{H_1}{H_0} = -\frac{1-\beta}{1+\beta} \frac{\Sigma_1}{\Sigma_0} + \frac{4-3\beta}{1+\beta} \frac{T_1}{T_0}, \quad (\text{A3})$$

$$\frac{p_1}{p_0} = \frac{2\beta}{1+\beta} \frac{\Sigma_1}{\Sigma_0} + \frac{4-3\beta}{1+\beta} \frac{T_1}{T_0}. \quad (\text{A4})$$

Note that equations (12) and (19) are adopted for the derivation.

Then the perturbative equation of the viscous heating (16) can be obtained from (1), (4), (A1), and (A3)-(A4) as

$$\frac{Q_{\text{vis},1}^+}{Q_{\text{vis},0}^+} = \frac{(4-3\beta)(1+\epsilon) - \gamma(3-4\beta) + \delta(8+\beta)}{1+\beta} \frac{T_1}{T_0} + \frac{-1+\beta-2\delta+2\gamma+2\beta\epsilon}{1+\beta} \frac{\Sigma_1}{\Sigma_0}, \quad (\text{A5})$$

When the electron scattering is the dominant source of the opacity, the radiative cooling (17) provides a linearized equation

$$\frac{Q_{\text{rad},1}^-}{Q_{\text{rad},0}^-} = 4 \frac{T_1}{T_0} - \frac{\Sigma_1}{\Sigma_0}. \quad (\text{A6})$$

By substituting equations (A5) and (A6) into equation (15) and using equation (A2), we obtain the instability condition

$$\frac{4-10\beta-7\gamma(1-\beta)+\delta(8+\beta)-(4-3\beta)(1-\gamma-\epsilon)}{1+\beta} > 0. \quad (\text{A7})$$

When we take $\epsilon = 1 - \gamma$, this equation reduces to equation (21) since $\beta \geq 0$.

When the free-free absorption mainly determines the opacity of the system, we can give a linearized equation for the radiative cooling, by using equations (11) and (A3),

$$\frac{Q_{\text{rad},1}^-}{Q_{\text{rad},0}^-} = \frac{23+9\beta}{2(1+\beta)} \frac{T_1}{T_0} - \frac{3+\beta}{1+\beta} \frac{\Sigma_1}{\Sigma_0}. \quad (\text{A8})$$

By substituting equations (A5) and (A8) into equation (15) and applying equation (A2), we obtain the instability criterion for the thermal instability,

$$\frac{7+21\beta+14\gamma(1-\beta)-2\delta(8+\beta)+2(4-3\beta)(1-\gamma-\epsilon)}{2(1+\beta)} < 0. \quad (\text{A9})$$

Note that this can reduce to the condition (23) when $\epsilon = 1 - \gamma$ since $\beta \geq 0$.

Finally we consider the stability condition of the system against the secular instability. When the secular instability grows, the negative diffusion amplifies the surface density so that the assumption (A2) cannot be adopted here. As mentioned in § 3.2, the viscous time scale, which is equivalent with the typical growth time of the secular instability, is much longer than the thermal time scale. The thermal equilibrium is thus maintained during the development of the secular instabilities, i.e., $Q_{\text{vis}}^+ = Q_{\text{rad}}^-$. This can provide a linearized equation

$$(-4+\gamma+4\delta)\frac{T_1}{T_0} + (1+\gamma-\delta)\frac{\Sigma_1}{\Sigma_0} + (1-\gamma+\delta)\frac{H_1}{H_0} + \epsilon\frac{p_1}{p_0} = 0, \quad (\text{A10})$$

for the case with the electron scattering as the main opacity source, and

$$(-15+2\gamma+8\delta)\frac{T_1}{T_0} + (4+2\gamma-2\delta)\frac{\Sigma_1}{\Sigma_0} + 2(-\gamma+\delta)\frac{H_1}{H_0} + 2\epsilon\frac{p_1}{p_0} = 0, \quad (\text{A11})$$

for the model with the free-free absorption as the dominant opacity source.

By combining equations (A3), (A4) and (A10) or (A11), we can express the quantities H_1/H_0 , T_1/T_0 and p_1/p_0 by Σ_1/Σ_0 for each model with different opacity source. Then by substituting these obtained equations into equation (18), we obtain the instability criterion for the secular instability.

When the electron scattering is the main opacity source, the instability condition is given as

$$\frac{5\gamma+4\epsilon+\beta(1+4\gamma+5\epsilon+\delta)}{4-10\beta-7\gamma(1-\beta)+\delta(8+\beta)-(4-3\beta)(1-\gamma-\epsilon)} > 0. \quad (\text{A12})$$

This condition is identical with equation (A7) when $\delta > 0$ since the numerator in equation (A12) has a positive value. Especially when $\epsilon = 1 - \gamma$, this equation reduces to equation (21).

The disks in which the free-free absorption becomes the dominant opacity source is unstable to the secular instability when

$$\frac{1+28\gamma+24\epsilon+2\delta+\beta(3+8\gamma+12\epsilon+2\delta)}{7+21\beta+14\gamma(1-\beta)-2\delta(8+\beta)+2(4-3\beta)(1-\gamma-\epsilon)} < 0. \quad (\text{A13})$$

Since the numerator of equation (A13) is positive when $\delta > 0$, the condition for the secular instability in the outer region is identical with that for the thermal instability given in equation (A9). When $\epsilon = 1 - \gamma$, this equation reduces to equation (23).

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